Quantum power in de Broglie–Bohm theory

2007 J. Phys. A: Math. Theor. 40 5155

(http://iopscience.iop.org/1751-8121/40/19/015)

J. Phys. A: Math. Theor. **40** (2007) 5155–5162 doi:10.1088/1751-8113/40/19/015

Quantum power in de Broglie–Bohm theory

Donald H Kobe

Department of Physics, University of North Texas, Denton, TX 76203-1427, USA

Received 30 June 2006, in final form 15 March 2007
Published 24 April 2007
Online at stacks.iop.org/JPhysA/40/5155

Abstract

In the de Broglie–Bohm approach to quantum mechanics for a charged particle in a time-dependent electromagnetic field the time derivative of the energy is equal to the classical power plus a quantum power. We show that the average of the quantum power is zero. The de Broglie–Bohm energy is obtained from the quantum mechanical energy operator, which is the Hamiltonian with the gauge-dependent scalar potential subtracted. The time derivative of the average de Broglie–Bohm energy is shown to be equal to Ehrenfest's theorem for the quantum energy operator.

PACS numbers: 03.65.Ta, 03.65.Sq, 03.65.Ca, 03.65.−w

1. Introduction

Shortly after the introduction of quantum mechanics by Heisenberg and Schrödinger, attempts were made by Madelung [1] and de Broglie [2] (see also [3] and references therein) to give it a causal interpretation in terms of trajectories. These attempts were superseded by the more popular Copenhagen interpretation that involves the collapse of the wavefunction. However, in the 1950s Bohm [4] and Takabayasi [5–8] revived this approach and addressed a number of criticisms of the theory. de Broglie [3, 9] also revisited his original approach. The book by Holland [10] in 1993 on the de Broglie–Bohm causal interpretation gave a new impetus to the subject. In the de Broglie–Bohm [11] approach to quantum mechanics particles follow 'quantum trajectories' that are different from the classical trajectories because of the addition of a quantum potential to the classical potentials present. These quantum trajectories have been called 'surrealistic' by Englert *et al* [12] because they appear nonintuitive. Nevertheless, the quantum trajectories have been useful in treating the tunnelling time problem by Leavens and co-workers [13]. The quantum trajectories have also been used to treat quantum chaos in a manner similar to classical chaos [14, 15].

In this paper we consider de Broglie–Bohm theory for the energy of a single, charged quantum particle in an external electromagnetic field. The de Broglie–Bohm energy is the sum of a kinetic energy term, the classical potential energy and a quantum potential. The quantum potential arises from the quantum kinetic energy and is therefore a 'fictitious' potential analogous to the centrifugal potential. The time derivative of the de Broglie–Bohm energy is equal to the classical power due to the nonconservative electric field plus a quantum power. If the de Broglie–Bohm average of the time derivative of the de Broglie–Bohm energy is taken, the result is shown to be equal to Ehrenfest's theorem for the energy operator in standard quantum mechanics. To derive this result we prove that the de Broglie–Bohm average of the quantum power is zero. The average of the time derivative of the de Broglie–Bohm energy is equal to the time derivative of its average. The expectation value of the quantum energy operator, the de Broglie–Bohm energy, and their time derivatives are all completely gauge invariant.

The extension of the de Broglie–Bohm approach to a charged particle in a time-dependent electromagnetic field is a significant generalization from previous treatments [10]. The most significant result in this paper is the proof that the de Broglie–Bohm average of the quantum power is zero. This result is the key to showing that the time derivative of the average de Broglie–Bohm energy is the same as Ehrenfest's theorem for the energy operator in standard quantum mechanics. The proper energy operator is the sum of the kinetic energy operator plus the conservative potential energy. This operator differs from the Hamiltonian by not including the gauge-dependent scalar potential of the time-dependent electromagnetic field [18–20]. The de Broglie–Bohm energy, including the quantum potential, is derived from this quantum mechanical energy operator. Ehrenfest's theorem for energy is also proved using this energy operator.

In section 2 we review the de Broglie–Bohm approach to quantum mechanics including the quantum Newton's second law. This quantum Newton's second law is used in section 3 to show that the time derivative of the de Broglie–Bohm energy is equal to the classical power due to the nonconservative electric field plus a quantum power. We then show in section 4 that the average of the quantum power is zero. The time derivative of the average de Broglie–Bohm energy is shown in section 5 to be equivalent to Ehrenfest's theorem for energy. Finally, the conclusion in section 6 also includes some discussion of the physical significance of the quantum power and quantum potential.

2. de Broglie–Bohm theory for charged particle in an electromagnetic field

We apply the de Broglie–Bohm formulation of quantum mechanics to a single charged particle in an external electromagnetic field [5, 10, 16]. We summarize the equations here to obtain the quantum Newton's second law and to establish the notation.

The Schrödinger equation for a particle of mass m and charge q in an external electromagnetic field characterized by the vector potential $\mathbf{A}(\mathbf{r}, t)$ and scalar potential $\phi(\mathbf{r}, t)$ with a wavefunction $\psi(\mathbf{r}, t)$ is

$$\hat{H}(\mathbf{r}, t)\psi = \left\{\frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 + q\phi + V\right\}\psi = \mathrm{i}\hbar\frac{\partial\psi}{\partial t}, \tag{1}$$

where the Hamiltonian operator is $\hat{H}(\mathbf{r}, t)$ and the potential energy $V = V(\mathbf{r})$ is conservative. The canonical momentum operator $\hat{\mathbf{p}} = -\mathrm{i}\hbar\nabla$ is conjugate to the coordinate $\mathbf{r}$.

In the de Broglie–Bohm approach the wavefunction ψ is written in the polar form $\psi = R\exp\{\mathrm{i}S/\hbar\}$ in terms of its modulus $R = R(\mathbf{r}, t)$ and phase $S/\hbar = S(\mathbf{r}, t)/\hbar$. We assume that the modulus and phase are at least twice differentiable in space and time. When this wavefunction is substituted into the Schrödinger equation (1) a complex equation involving R and S is obtained.

The real part of the complex equation resulting from the Schrödinger equation is a quantum Hamilton–Jacobi equation,

$$\frac{1}{2m}(\nabla S - q\mathbf{A})^2 + q\phi + V + Q + \frac{\partial S}{\partial t} = 0. \tag{2}$$

The quantum Hamilton–Jacobi equation differs from the classical Hamilton–Jacobi equation [17] by the addition of a *quantum potential* $Q = Q(\mathbf{r}, t)$ defined as [4]

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{3}$$

which in general is nonconservative. Since the quantum potential originates from the quantum kinetic energy, it is a 'fictitious potential' in the same sense as the centrifugal potential.

The first term in equation (2) is the de Broglie–Bohm kinetic energy, since the de Broglie–Bohm mechanical momentum $m\mathbf{v}$ is defined as

$$m\mathbf{v} = \nabla S(\mathbf{r}, t) - q\mathbf{A}(\mathbf{r}, t), \tag{4}$$

where $\nabla S = \mathbf{p}$ is the canonical momentum in Hamilton–Jacobi theory [17]. Under a gauge transformation the velocity is invariant. The coordinate $\mathbf{r}$ in R and S is now interpreted as a quantum trajectory $\mathbf{r} = \mathbf{r}(t)$. The velocity is therefore $\mathbf{v} = \dot{\mathbf{r}}(t)$, so equation (4) gives a set of coupled first-order differential equations for the quantum trajectory $\mathbf{r} = \mathbf{r}(t)$. When an initial position $\mathbf{r}(0)$ is specified, their solution is a quantum trajectory $\mathbf{r}(t)$ for the particle. This approach to quantum trajectories is called the *minimal* de Broglie–Bohm theory [10].

The imaginary part of the complex equation resulting from the Schrödinger equation is, after some rearranging, the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0, \tag{5}$$

for probability conservation. The probability density is $\rho = \rho(\mathbf{r}, t) = R^2$ and the current density is $\mathbf{J} = \mathbf{J}(\mathbf{r}, t) = \mathbf{v}\rho = \dot{\mathbf{r}}\rho$.

Taking the negative gradient of the quantum Hamilton–Jacobi equation (2) and using the hydrodynamic derivative $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{\mathbf{r}} \cdot \nabla - \frac{\partial}{\partial t}$, we obtain a quantum Newton's second law [5, 7, 10]

$$m\ddot{\mathbf{r}}(t) = q\mathbf{E}(\mathbf{r}, t) + q\dot{\mathbf{r}}(t) \times \mathbf{B}(\mathbf{r}, t) - \nabla V - \nabla Q. \tag{6}$$

The first and second terms on the right-hand side of equation (6) are the electric and Lorentz forces, respectively, where the electric field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ are

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}, \tag{7}$$

in terms of the scalar and vector potentials. The third term is the conservative force $-\nabla V(\mathbf{r})$ due to the conservative potential energy in the Schrödinger equation (1). The last term in equation (6) is the quantum force $-\nabla Q$, which in general is nonconservative because it depends on time. Except for the quantum force, equation (6) is the classical Newton's second law.

3. Quantum energy and power

The energy of a charged particle in an external time-dependent electromagnetic field is not conserved, but its time rate of change is equal to the power exchanged with the field. Here we find the de Broglie–Bohm energy, which is time dependent because some forces are nonconservative [18–20].

The scalar product of the velocity $\mathbf{v}$ with the quantum Newton's second law (6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\mathbf{v}^2 + V\right) = \mathbf{v} \cdot q\mathbf{E}(\mathbf{r}, t) - \mathbf{v} \cdot \nabla Q, \tag{8}$$

since the magnetic field does no work. The potential V is conservative so $\mathrm{d}V/\mathrm{d}t = \dot{\mathbf{r}} \cdot \nabla V$. The first term on the right-hand side is the power due to the nonconservative electric field and the second term is due to the quantum force.

Using the definition of the total time derivative of the quantum potential, we can write the second term on the right-hand side of equation (8) as

$$-\dot{\mathbf{r}} \cdot \nabla Q = -\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\partial Q}{\partial t}, \tag{9}$$

since $\mathbf{v} = \dot{\mathbf{r}}$. Substituting equation (9) into equation (8) and rearranging, we obtain the total de Broglie–Bohm energy

$$\mathcal{E} = \tfrac{1}{2} m \mathbf{v}^2 + Q + V, \tag{10}$$

that includes the quantum potential Q. The time-dependent scalar potential $\phi(\mathbf{r}, t)$ is not included in equation (10) because $\phi(\mathbf{r}, t)$ is nonconservative and gauge dependent. Therefore the de Broglie–Bohm energy is gauge invariant (up to a constant) [19]. The quantum Hamilton–Jacobi equation (2) shows that the energy in equation (10) can also be written as $\mathcal{E} = -\partial S/\partial t - q\phi$, where the right-hand side is gauge invariant [6].

From equations (8)–(10) the time derivative of the energy is

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} = \mathcal{P}, \tag{11}$$

where the total power $\mathcal{P}$ is

$$\mathcal{P} = \mathbf{v} \cdot q\mathbf{E}(\mathbf{r}, t) + \frac{\partial Q}{\partial t}. \tag{12}$$

The first term on the right-hand side is the classical power due to the nonconservative electric field. The second term $\partial Q/\partial t$ is defined as the *quantum power* [10]. The quantum power in equation (12) shows the deviation of the quantum system from the classical power. In the next section we show that the average of the quantum power is zero, so the deviation is both positive and negative. In other words, the system borrows energy or gives up energy per unit time to an unspecified source or sink, which Holland [10] identifies as the quantum field.

If the classical power due to the time-dependent electric field and the quantum power are both zero in equation (12), the energy $\mathcal{E}$ in equation (10) is conserved. In this case the quantum potential Q has no explicit time dependence, so the magnitude of the wavefunction is also time independent and the system is described by a stationary state wavefunction. For a stationary state, the wavefunction may have classically allowed regions and classically forbidden regions. Since the quantum potential is the total kinetic energy for a stationary state with a phase $S = S(t)$ only, $Q > 0$ is positive in a classically allowed region and $Q < 0$ is negative in a classically forbidden region. For a one-dimensional system the definition of quantum potential in equation (3) shows that the second derivative (with respect to spatial variable x) of the modulus $R''(x, t) < 0$ is negative in an allowed region and positive $R''(x, t) > 0$ in a forbidden region. In other words, the curvature of the modulus is down in an allowed region and up in a forbidden region, where the wavefunction is damped. Thus the quantum potential gives some insight into the qualitative behaviour of the system.

4. Average quantum power and energy

We now take the de Broglie–Bohm average of equation (11). We first prove that the average quantum power is zero. Then we show that the de Broglie–Bohm average of the time derivative of the energy is equal to the time derivative of the average energy.

The de Broglie–Bohm average of equation (11) is

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \right\rangle_B = q\,\langle \mathbf{v}\cdot\mathbf{E}\rangle_B + \left\langle \frac{\partial Q}{\partial t} \right\rangle_B, \tag{13}$$

where the de Broglie–Bohm average is $\langle \cdots \rangle_B = \int \mathrm{d}^3 r\,\rho(\cdots)$ and the probability density $\rho = \rho(\mathbf{r}, t) = R^2$ is in general time dependent.

Using integration by parts, we show that the de Broglie–Bohm average of the quantum power is zero:

$$\begin{aligned}\left\langle \frac{\partial Q}{\partial t} \right\rangle_B &= -\frac{\hbar^2}{2m}\int \mathrm{d}^3 r\, R^2 \frac{\partial}{\partial t}\left(\frac{\nabla^2 R}{R}\right) \\ &= -\frac{\hbar^2}{2m}\int \mathrm{d}^3 r \left(R\nabla^2\frac{\partial R}{\partial t} - \frac{\partial R}{\partial t}\nabla^2 R\right) \\ &= -\frac{\hbar^2}{2m}\int \mathrm{d}^3 r\, \nabla\cdot\left(R\nabla\frac{\partial R}{\partial t} - \frac{\partial R}{\partial t}\nabla R\right) \\ &= 0,\end{aligned} \tag{14}$$

because the boundary condition is $r^2 R \to 0$ as $r \to \infty$.

Using the equation of continuity (5), we now show that the time derivative of the de Broglie–Bohm average energy is equal to the average of the time derivative of the energy. Since $\mathbf{J} = \mathbf{v}\rho$, the time derivative of the de Broglie–Bohm average energy is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{E}\rangle_B &= \int \mathrm{d}^3 r \left(\frac{\partial \rho}{\partial t}\mathcal{E} + \rho\frac{\partial \mathcal{E}}{\partial t}\right) \\ &= \int \mathrm{d}^3 r \left(-\mathcal{E}\nabla\cdot\mathbf{J} + \rho\frac{\partial \mathcal{E}}{\partial t}\right) \\ &= \int \mathrm{d}^3 r \left(\rho\mathbf{v}\cdot\nabla\mathcal{E} + \rho\frac{\partial \mathcal{E}}{\partial t}\right) \\ &= \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} \right\rangle_B,\end{aligned} \tag{15}$$

from integration by parts because of the boundary condition. This relation holds for the time derivative of the de Broglie–Bohm average of any quantity because of the equation of continuity [10].

When equations (14) and (15) are substituted into equation (13) we obtain the time derivative of the de Broglie–Bohm average energy

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{E}\rangle_B = q\langle \mathbf{v}\cdot\mathbf{E}\rangle_B, \tag{16}$$

where the right-hand side is the average of the classical power. Equation (16) is the same form as the classical power for a particle in an electromagnetic field. It is also reminiscent of Ehrenfest's theorem for the energy operator in standard quantum mechanics.

5. Ehrenfest's theorem for energy

Ehrenfest's theorem in standard quantum mechanics uses the expectation values of quantum operators. If the energy operator is used, we obtain the Ehrenfest theorem for energy. We now show the Ehrenfest theorem for energy is equivalent to equation (16) in de Broglie–Bohm theory.

The energy of a particle should be gauge invariant (up to a constant), so it is not in general equal to the Hamiltonian [19]. The Hamiltonian, which depends on the gauge, is a Legendre transformation on the Lagrangian to express it in terms of the canonical momentum. On the other hand, the energy operator $\hat{E}$ is the sum of kinetic energy operator and conservative potential energy. It is equal to the Hamiltonian operator in equation (1) minus the time-dependent scalar potential [18–20]

$$\hat{E} = \frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 + V(\mathbf{r}), \tag{17}$$

where $m\hat{\mathbf{v}} = \hat{\mathbf{p}} - q\mathbf{A}$ is the gauge-invariant mechanical momentum operator and $\hat{\mathbf{p}} = -\mathrm{i}\hbar\nabla$ is the canonical momentum operator. Ehrenfest's theorem for the energy is [18]

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{E}\rangle_\psi = \langle\hat{P}\rangle_\psi, \tag{18}$$

where $\hat{P}$ is the quantum power operator. The quantum expectation value is defined as usual to be $\langle\cdots\rangle_\psi = \int \mathrm{d}^3r\, \psi^*(\cdots)\psi$. The power operator $\hat{P}$ on the right-hand side of equation (18) is [18–20]

$$\hat{P} = \tfrac{1}{2}q\{\hat{\mathbf{v}}; \mathbf{E}\}, \tag{19}$$

due to the time-dependent electric field $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$. The power operator is Hermitian because the anticommutator of the dot product between two vector operators $\hat{\mathbf{Y}}$ and $\hat{\mathbf{Z}}$ is defined to be the Hermitian operator $\{\hat{\mathbf{Y}}; \hat{\mathbf{Z}}\} = \hat{\mathbf{Y}} \cdot \hat{\mathbf{Z}} + \hat{\mathbf{Y}} \cdot \hat{\mathbf{Z}}$.

The expectation value of the Hermitian energy operator in equation (17) is real and equal to the de Broglie–Bohm average of the energy in equation (10),

$$\begin{aligned}
\langle\hat{E}\rangle_\psi &= \int \mathrm{d}^3r\, \psi^* \left(\frac{1}{2m}(\hat{\mathbf{p}} - q\mathbf{A})^2 + V\right) R\exp(\mathrm{i}S/\hbar) \\
&= \int \mathrm{d}^3r\, \rho\, \mathrm{Re}\left(\frac{1}{2m}[(-\mathrm{i}\hbar R^{-1}\nabla R + \nabla S - q\mathbf{A})^2 - \hbar^2\nabla\cdot(R^{-1}\nabla R)] + V\right) \\
&= \int \mathrm{d}^3r\, \rho \left(\frac{1}{2}m\mathbf{v}^2 + Q + V\right) = \langle\mathcal{E}\rangle_B\,,
\end{aligned} \tag{20}$$

where the quantum potential Q is given in equation (3), the de Broglie–Bohm mechanical momentum is $m\mathbf{v} = \nabla S - q\mathbf{A}$ and the wavefunction in polar form is $\psi = R\exp(\mathrm{i}S/\hbar)$. From equation (20) we see that the quantum potential originates from the quantum kinetic energy operator, which justifies it being called a 'fictitious' potential.

The expectation value of the Hermitian power operator in equation (18) is equal to the de Broglie–Bohm average of the power in equation (16),

$$\begin{aligned}
\frac{q}{2}\langle\{\hat{\mathbf{v}}; \mathbf{E}\}\rangle_\psi &= \frac{q}{2}\int \mathrm{d}^3r\, \psi^*\{\hat{\mathbf{v}}; \mathbf{E}\}\psi \\
&= \frac{q}{m}\mathrm{Re}\int \mathrm{d}^3r\, \psi^*\mathbf{E}\cdot(\hat{\mathbf{p}} - q\mathbf{A})R\exp(\mathrm{i}S/\hbar) \\
&= \frac{q}{m}\int \mathrm{d}^3r\, \rho\mathbf{E}\cdot(\nabla S - q\mathbf{A}) \\
&= q\,\langle\mathbf{v}\cdot\mathbf{E}\rangle_B\,,
\end{aligned} \tag{21}$$

where the de Broglie–Bohm mechanical momentum is $m\mathbf{v} = \nabla S - q\mathbf{A}$.

Therefore, when equations (20) and (21) are substituted into Ehrenfest's theorem (18), we obtain the time rate of change of the de Broglie–Bohm average energy $\frac{\mathrm{d}}{\mathrm{d}t}\langle\mathcal{E}\rangle_B = q\,\langle\mathbf{v}\cdot\mathbf{E}\rangle_B$ in equation (16). In general, all the Ehrenfest theorems are in one-to-one correspondence with the corresponding de Broglie–Bohm average equations.

6. Conclusion

The de Broglie–Bohm theory offers an alternative interpretation to the usual interpretation of quantum mechanics based on the collapse of the wavefunction. de Broglie–Bohm theory gives a Hamilton–Jacobi equation with a quantum potential, so it is interpreted in terms of quantum trajectories that differ from the classical ones because of the effect of the quantum force. An equation of continuity for probability conservation is also obtained. Since only averages are observable, the resulting theory is similar to classical statistical mechanics.

The de Broglie–Bohm energy of a quantum particle in a time-dependent electromagnetic field is the sum of a de Broglie–Bohm kinetic energy, a quantum potential and a conservative potential energy. The time derivative of this energy is equal to the power due to a nonconservative electric field plus a quantum power. The average power in de Broglie–Bohm theory is equal to the expectation value of the power operator in standard quantum theory. We show that the de Broglie–Bohm average of the quantum power term is zero, so the time derivative of the de Broglie–Bohm average energy is equivalent to Ehrenfest's theorem for energy. Since the de Broglie–Bohm theory is an interpretation of quantum mechanics, the equivalence of the de Broglie–Bohm average power with the expectation value of the power operator in standard quantum mechanics is reassuring.

The quantum power $\partial Q/\partial t$ indicates how much the quantum system deviates from the system with only a classical power term. The average quantum power is zero, so the deviation is both positive and negative. An external source or sink of energy is required for this deviation that may be attributed to the quantum field. The de Broglie–Bohm theory is not completely satisfactory in accounting for this lack of energy conservation [10].

For a stationary state in one dimension, the quantum potential is the only contribution to the kinetic energy since $p = 0$. If the system is in a classically allowed or forbidden region, the quantum potential Q will be positive or negative, respectively. The corresponding curvature of the modulus of the wavefunction R will be negative or positive, indicating downward or upward curvature, respectively. In a classically forbidden region, the wavefunction would be expected to have an upward curvature indicating that it is decreasing rapidly. Thus, in the case of a stationary state the quantum potential is useful in giving a qualitative understanding of the system.

Acknowledgments

I would like to thank Professor W P Schleich and Professor Carlos Ordonez for discussions and Professor M O Scully for his encouragement. This work is partially supported by a grant from ONR N00014-03-1-0639/TAMU TEES 53494.

References

[1] Madelung E 1926 *Zeit. f. Phys.* **40** 322
[2] de Broglie L 1926 *Nature* **118** 441
[3] de Broglie L 1960 *Non-Linear Wave Mechanics: A Causal Interpretation* (Amsterdam: Elsevier)
[4] Bohm D 1952 *Phys. Rev.* **85** 166
Bohm D 1952 *Phys. Rev.* **85** 180
Wheeler J A and Zurek W H (ed) 1983 *Quantum Theory of Measurement* (Princeton: Princeton University Press) pp 369–396 (reprinted in)
[5] Takabayasi T 1952 *Prog. Theor. Phys.* **8** 143
[6] Takabayasi T 1953 *Prog. Theor. Phys.* **9** 187
[7] Takabayasi T 1955 *Prog. Theor. Phys.* **14** 283

[8] Takabayasi T 1983 *Prog. Theor. Phys.* **69** 1323
[9] de Broglie L 1964 *The Current Interpretation of Wave Mechanics: A Critical Study* ed with a chapter by J Andrade e Silva (Amsterdam: Elsevier) chapter VIII
[10] Holland P R 1993 *The Quantum Theory of Motion—an Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics* (Cambridge: Cambridge University Press) pp 117–20
[11] Bohm D and Hiley B J 1987 *Phys. Rep.* **144** 323
[12] Englert B-G, Scully M O, Sussmann G and Walther H 1992 *Zeit. f. Naturforschung* A **47** 1175
[13] Leavens C R 2002 Bohm trajectory approach to timing electrons *Time in Quantum Mechanics* ed J G Muga, R S Mayato and I L Egusquiza (Berlin: Springer) pp 121–52
[14] Schwengelbeck U and Faisal F H M 1995 *Phys. Lett.* A **199** 281
[15] Zheng Y D and Kobe D H 2006 *Chaos Solitons Fractals* ScienceDirect.com doi:10.1016/j.chaos.2006.06.045
[16] Schwengelbeck U and Faisal F H M 1995 *Phys. Lett.* A **207** 31
[17] Goldstein H, Poole C and Safko J 2002 *Classical Mechanics* 3rd edn (San Francisco: Addison-Wesley) chapter 10
[18] Yang K-H 1976 *Ann. Phys., NY* **101** 62
[19] Kobe D H and Yang K-H 1987 *Eur. J. Phys.* **8** 236
[20] Kobe D H, Wen E C-T and Yang K-H 1982 *Phys. Rev.* D **26** 1927